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Generating Functions for $P_1^r(n)$ and $P_2^r(n)$

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Abstract

In 1970 George E. Andrews defined the generating functions for $P_1{}^r(n)$ and $P_2{}^r(n)$. In this article these generating functions are discussed elaborately. This paper shows how to prove the theorem $P_2{}^r(n)=P_3{}^r(n)$ with a numerical example when *n* = 9 and *r* = 2. In 1966 Andrews defined the terms $A'(n)$ and $B'(n)$, but this paper proves the remark $A'(n)=B'(n)$ with the help of an example when *n* = 10. In 1961 N. Bourbaki defined the term $P(n,m)$. This paper shows how to prove a Remark in terms of $P(n,m)$, where $P(n,m)$ is the number of partitions of the type of enumerated by $P_3{}^r(n)$ with the further restrictions that $b_1 \le 2m$.

Keywords: Generating functions, number of partitions.

1 Introduction

We give definitions of $P_1{}^r(n)$, $P_2{}^r(n)$, $P_3{}^r(n)$, $A'(n)$, $B'(n)$, and $P(n,m)$. Then we generate the function for $P_1{}^r(n)$, $P_2{}^r(n)$, and $P_3{}^r(n)$, which are collected from George E. Andrews [1] and Hardy and Wright [5] and prove the Theorem $P_2{}^r(n)=P_3{}^r(n)$. George E. Andrews [1] has already prove the remark $P_1{}^r(n)=P_3{}^r(n)$ and we give a numerical example for $A'(n)=B'(n)$. Finally we prove a remark which is related to the term $P(n,m)$.

2 Definitions

$P_1{}^r(n)$: The number of partitions of *n* into part that are either even or not congruent to $4r-2(\bmod 4r)$ or odd and congruent to $2r-1, 4r-1(\bmod 4r)$ [2].

$P_2{}^r(n)$: The number of partitions of *n* into parts that are either even or else congruent to $2r-1(\bmod 2r)$ with the further restriction that only even parts may be repeated.

$P_3{}^r(n)$: The number of partitions of *n* of the form $n=b_1+b_2+...+b_s$, where $b_i \ge b_{i+1}$, and for b_i odd, $b_i-b_{i+1}\ge 2r-1$ $(1\le i\le s,$ where $b_{s+1}=0)$.

$A'(n)$: The number of partitions of *n* into parts congruent to 0, 2, 3, 4, 7 (mod 8).

$B'(n)$: The number of partitions of *n* of the form $n=b_1+b_2+...+b_s$, where $b_s\ge 2$, $b_i\ge b_{i+1}$, and if b_i is odd, $b_i-b_{i+1}\ge 3$.

$P(n,m)$: The number of partitions of the type enumerated by $P_3{}^r(n)$, with the further restriction that $b_1\le 2m$ [4].

3 Consider the Generating Function with r ≥ 2, [3]

We have,

$$\prod_{j=1}^{\infty}\left(1-x^{4rj-2}\right)\left(1-x^{2j}\right)^{-1}\left(1-x^{2rj-1}\right)^{-1} \quad (1)$$

$$=\prod_{j=1}^{\infty}\left(1-x^{4rj-2}\right)\left(1+x^{2j}+x^{4j}+...\infty\right)\times \left(1+x^{2rj-1}+x^{4rj-2}+...\infty\right)$$

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$$= 1 + \sum_{n=1}^{\infty} P_1^{\,r}(n)\, x^n \ ,$$

where the coefficient $P_1^{\,r}(n)$ is the number of partitions of n into parts that are either even and not congruent to $4r-2(\bmod 4r)$ or odd and congruent to $2r-1, 4r-1(\bmod 4r)$.

We consider a function, which is of the form;

$$\prod_{j=1}^{\infty}\left(1+x^{2rj-1}\right)\left(1-x^{2j}\right)^{-1}; r \geq 2$$

$$= \prod_{j=1}^{\infty}\left(1+x^{2rj-1}\right)\left(1+x^{2j}+x^{4j}+\ldots\infty\right)$$

$$= 1 + \sum_{n=1}^{\infty} P_2^r(n)\, x^n \qquad (2)$$

where the coefficient $P_2^r(n)$ is the number of partitions of n into parts that are either even or else congruent to $2r-1(\bmod 2r)$ with the further restriction that only even parts may be repeated.

From (1) we have;

$$1+\sum_{n=1}^{\infty} P_1^r(n)\, x^n = \prod_{j=1}^{\infty}\left(1-x^{4rj-2}\right)\left(1-x^{2j}\right)^{-1}\left(1-x^{2rj-1}\right)^{-1}$$

$$= \prod_{j=1}^{\infty}\frac{\left(1+x^{2rj-1}\right)}{\left(1-x^{2j}\right)} = 1 + \sum_{n=1}^{\infty} P_2^r(n)\, x^n \ , \text{ by (2).}$$

Now equating the coefficient of x^n from the both sides we get;

$$P_1^r(n) = P_2^r(n).$$

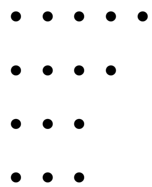
Here we give a Theorem, which is related to the terms $P_2^r(n)$ and $P_3^r(n)$.

Theorem: Let $r \geq 2$ be an integer. Let $P_2^r(n)$ denote the number of partitions of n into parts that are either even or else congruent to $2r-1(\bmod 2r)$ with the further restriction that only even parts may be repeated. Let $P_3^r(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + \ldots + b_s$, where $b_i \geq b_{i+1}$, and for b_i odd, $b_i - b_{i+1} \geq 2r-1$ $\left(1 \leq i \leq s, \text{ where } b_{s+1} = 0\right)$.

Then $P_2^r(n) = P_3^r(n)$.

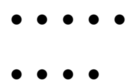
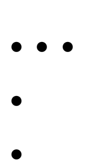
Proof: Let Ψ' be a partition of the type enumerated by $P_3^r(n)$. We represent Ψ' graphically with each even part $2m$ represented by two rows of m nodes and each odd part $2m + 1$ represented by two rows of m+1 nodes and m nodes respectively.

Such as 9 + 6 becomes;



Now we may consider the graph vertically with the condition that r columns are always to be grouped as a single part, whenever the lowest node in the most right hand column of the group is not presented there. If r = 2, we obtain in this manner, the partition 4 +4 + 4 + 3. Now since the condition on partitions enumerated by $P_3^r(n)$ is $b_i - b_{i+1} \geq 2r-1$, whenever b_i is odd, we see that our grouping of r columns always have one less node that a rectangle of $r-2v$ nodes, when v is any positive integer. Thus a part congruent to $2r-1(\bmod 2r)$ is produced. Since originally odd parts were distinct, we see that now odd parts will be congruent to $2r-1(\bmod 2r)$ and will not be repeated and since originally all odd parts were greater or equal to $2r-1$, we see that there will always be r columns available for each grouping. Thus in this case we have produced a partition of the type enumerated by $P_2^r(n)$. Clearly our correspondence is one to one, however, the above process is reversible and thus the correspondence is onto. So that $P_2^r(n) = P_3^r(n)$. Hence the Theorem.

Example 1: We take r = 2, n = 9. The corresponding partitions are listed opposite each other in the following table:

$P_3^r(9)$		$P_2^r(9)$
9 with relevant graph		2 + 2 + 2 + 3
7 + 2 ” ” ”		4 + 2 + 3

6 + 3 ” ” ” • • • 7 + 2

• • •

• •

•

5 + 2 + 2 ” ” ” • • • 6 + 3

• •

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Now we can write $P_3^r(9) = P_2^r(9) = 4$. Here we give some remarks.

Remark 1: $P_1^r(n) = P_3^r(n)$, if $r \geq 2$ i.e., let $r \geq 2$ be an integer. Let $P_1^r(n)$ denote the number of partitions of n into parts that are either even and not congruent to $4r - 2(\mathrm{mod}\,4r)$ or odd and congruent to $2r-1, 4r-1(\mathrm{mod}\,4r)$. Let $P_3^r(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + ... + b_s$, where $b_i \geq b_{i+1}$, and for b_i odd, $b_i - b_{i+1} \geq 2r - 1$ $(1 \leq i \leq s, \text{where}\, b_{s+1} = 0)$.

Then $P_1^r(n) = P_3^r(n)$. It is proved in George E. Andrews [1]. We can establish the following Remark:

Remark 2: Let $A'(n)$ denote the number of partitions of n into parts congruent to 0, 2, 3, 4, 7 (mod 8). Let $B'(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + ... + b_s$, where $b_i \geq b_{i+1}$, and for b_i is odd, $b_i - b_{i+1} \geq 3$. Then $A'(n) = B'(n)$.

Example 2: If n = 10, the eight partitions enumerated by $A'(10)$ are 10, 8 + 2, 7 + 3, 4 + 4 + 2, 4 + 2 + 2 + 2, 4 + 3 + 3, 3 + 3 + 2 + 2, 2+ 2 + 2 + 2 + 2.

The eight partitions enumerated by $B'(10)$ are 10, 8 + 2, 7 + 3, 6 + 4, 6 + 2 + 2, 4 + 4 + 2, 4 + 2 + 2 + 2, 2 + 2 + 2 + 2 + 2.

Thus $A'(10) = B'(10)$.

Here we give the Remark which is related to the term $P(n,m)$.

Remark 3:
$P(n,m) - P(n,m-1) = P(n-2m,m) + P(n-2m+1,m-r)$, where $P(n,m)$ is the number of partitions of the type enumerated by $P_3^r(n)$ with the further restriction that $b_1 \leq 2m$.

Proof: Here $P(n,m) - P(n,m-1)$ denotes the number of partitions of the type enumerated by $P(n,m)$ with the further restriction that either 2m or 2m–1 is the largest part. If 2m is the largest part, we remove it. We obtain a partition of the type enumerated by $P(n-2m,m)$. If 2m– 1 is the largest part, then the next largest part does not exceed 2m–1–(2r–1) or 2m–2r, since 2m–1 is an odd part. If 2m–1 is removed from the partition under consideration, we obtain a partition of the type enumerated by $P(n-2m+1,m-r)$. Hence the above process establishes a (1, 1) correspondence between those partitions enumerated by $P(n,m) - P(n,m-1)$ and the totality of partitions, which are enumerated either by

$P(n-2m,m)$ or by $P(n-2m+1,m-r)$.

Thus,

$$P(n,m) - P(n,m-1) = P(n-2m,m) + P(n-2m+1,m-r).$$

Hence the Remark.

4 Conclusions

We have seen that for any positive integer of n and $r \geq 2$ the Theorem $P_1^r(n) = P_3^r(n)$ is satisfied. We have shown the Theorem $A'(n) = B'(n)$ is true with the help of example when n = 10.

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