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Mock Theta Conjectures

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Abstract

This paper shows how to prove the two Theorems first and second mock theta conjectures respectively.

Keywords: Mock theta, rank of partition.

1. Introduction

We give the definitions of π, rank of partition, $N(m,n)$, $N(m,t,n)$, $\rho_0(n)$, $\rho_1(n)$, z, $(x)_\infty$, $(zx)_\infty$, $(x^n)_m$, $(x^k;x^5)_m$ which are collected from Partitions Yesterday and Today [4], Generalizations of Dyson's Rank [3], Ramanujan's Lost Notebook [2]. We generate the generating functions for $\rho_0(n)$, and $\rho_1(n)$ [2] and prove the two Theorems first and second mock theta conjectures respectively. Finally we give two numerical examples which are related to first and second mock theta conjectures respectively when $n = 1$.

2. Definitions

π : A partition.

Rank of partition: The largest part of a partition π minus the number of parts of π.

$N(m,n)$: The number of partitions of *n* with rank *m*.

$N(m,t,n)$: The number of partition of *n* with rank congruent to *m* modulo *t*.

$\rho_0(n)$: The number of partitions of *n* with unique smallest part and all other parts ≤ the double of the smallest part.

$\rho_1(n)$: The number of partitions of *n* with unique smallest part and all other parts ≤ one plus the double of the smallest part.

z : The set of complex numbers.

$(x)_\infty$: The product of infinite factors is defined as follows:

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)\ldots\infty .$$

$(zx)_\infty$: The product of infinite factors is defined as follows:

$$(zx)_\infty = (1-zx)(1-zx^2)(1-zx^3)\ldots\infty .$$

$(x^n)_m$: The product of *m* factors is defined as follows:

$$(x^n)_m = (1-x^n)(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{n+m-1}).$$

$(x^k;x^5)_m$: The product of *m* factors is defined as follows:

$$(x^k;x^5)_m = (1-x^k)(1-x^{k+5})(1-x^{k+10})\ldots(1-x^{k+(m-1)5}).$$

3. Mock Theta Functions (2)

We quote the relations below [1, 2]:

$$F(x) = \frac{(1-x)(1-x^2)(1-x^3)\ldots\infty}{(1-2x\cos\frac{2n\pi}{5}+x^2)(1-2x^2\cos\frac{2n\pi}{5}+x^4)\ldots\infty} .$$

$$f^{/}(x) = 1+\frac{x}{1-2x\cos\frac{2n\pi}{5}+x^2}+\frac{x^4}{(1-2x\cos\frac{2n\pi}{5}+x^2)(1-2x^2\cos\frac{2n\pi}{5}+x^4)}+\ldots\infty,$$

$n = 1$ or 2.

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$$F(x^{\frac{1}{5}}) = A(x) - 4x^{\frac{1}{5}}\cos\frac{2n\pi}{5}B(x) + 2x^{\frac{2}{5}}\cos\frac{4n\pi}{5}C(x) -$$

$$2x^{\frac{3}{5}}\cos\frac{2n\pi}{5}D(x). \qquad (1)$$

$$f'(x^{\frac{1}{5}}) = \left\{A(x) - 4\sin^2\frac{2n\pi}{5}\Phi(x)\right\} + x^{\frac{1}{5}}B(x) + 2x^{\frac{2}{5}}\cos\frac{2n\pi}{5}C(x) -$$

$$2x^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(x) + 4\sin^2\frac{2n\pi}{5}.\frac{\psi(x)}{x}\right\}. \qquad (2)$$

$$A(x) = \frac{1 - x^2 - x^3 + x^9 + \ldots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\ldots\infty}$$

$$B(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots\infty}{(1-x)(1-x^4)(1-x^6)\ldots\infty}$$

$$C(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots\infty}{(1-x^2)(1-x^3)(1-x^7)\ldots\infty}$$

$$D(x) = \frac{1 - x - x^4 + x^7 + \ldots\infty}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\ldots\infty}$$

$$\phi(x) = -1 + \left\{\frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \right.$$

$$\left.\frac{x^{20}}{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})} + \ldots\infty\right\},$$

But we get;

$$A(x^5) - 4x\cos\frac{2\pi}{5}B(x^5) + 2x^2\cos\frac{4\pi}{5}C(x^5) -$$

$$2x^3\cos\frac{2\pi}{5}D(x^5)$$

$$= 1 - 4x\cos^2\frac{2\pi}{5} + 2x^2\cos\frac{4\pi}{5} - 2x^3\cos\frac{2\pi}{5} + 2x^5 -$$

$$4x^6\cos^2\frac{2\pi}{5} + 2x^8\cos\frac{2\pi}{5} - x^{10} + \ldots\infty$$

$$\Psi(x) = -1 + \left\{\frac{1}{1-x^2} + \frac{x^5}{(1-x^2)(1-x^3)(1-x^7)} + \right.$$

$$\left.\frac{x^{20}}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})} + \ldots\infty\right\}.$$

Now,

$$\frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \ldots\infty$$

$$= 3\phi(x) + 1 - A(x).$$

And,

$$\frac{x}{1-x} + \frac{x^2}{(1-x^2)(1-x^3)} + \frac{x^3}{(1-x^3)(1-x^4)(1-x^5)} + \ldots\infty$$

$$= 3\Psi(x) + xD(x).$$

We assume without loss of generality that $n = 1$. Let $\zeta = \exp^{\frac{2\pi i}{5}}$, then we may write the definitions of $F(x)$ and $f'(x)$ as;

$$F(x) = \frac{(x)_\infty}{(\zeta x)_\infty(\zeta^{-1}x)_\infty}$$

and

$$f'(x) = 1 + \sum_{n=1}^{\infty}\frac{x^{n^2}}{(1-\zeta x)(1-\zeta^{-1}x)\ldots(1-\zeta x^n)(1-\zeta^{-1}x^n)}$$

$$= \sum_{n=1}^{\infty}\frac{x^{n^2}}{(\zeta x)_n(\zeta^{-1}x)_n},$$

where we have used the relations;

$(a)_0 = 1$, $(a)_n = (1-a)(1-ax)\ldots(1-ax^{n-1})$, for $n \geq 1$

and

$$(a)_\infty = \lim_{n\to\infty}(a)_n = \prod_{n=1}^{\infty}(1-ax^{n-1}).$$

After replacing x by x^5 we see that (1) and (2) are identities for $F(x)$ and $f'(x)$. We note that the numerators in the definitions of $A(x)$ and $D(x)$ are theta series in x and hence may be written as infinite products using Jecobi's triple product identity;

$$\prod_{n=1}^{\infty}(1 - zx^n)(1 - z^{-1}x^{n-1})(1 - x^n)$$

$$= \prod_{n=-\infty}^{\infty}(-1)^n z^n x^{\frac{n(n+1)}{2}} \qquad (3)$$

$$= \ldots + z^{-2}x - z^{-1} + 1 - zx + z^2x^3 - \ldots\infty .$$

where $z \neq 0$ and $|x| < 1$.

Replacing x by x^5 and z by x^{-3} we get from (3);

$$\prod_{n=1}^{\infty}\left(1 - x^{5n-3}\right)\left(1 - x^{5n-2}\right)\left(1 - x^{5n}\right)$$

$$= \ldots + x^{11} + 1 - x^2 + x^9 - \ldots\infty$$

$$= 1 - x^2 - x^3 + x^9 + x^{11} - \ldots\infty .$$

Again replacing x by x^5 and z by x^{-3} (3) becomes;

$$\prod_{n=1}^{\infty}\left(1 - x^{5n-4}\right)\left(1 - x^{5n-1}\right)\left(1 - x^{5n}\right)$$

$$= \ldots + x^{13} - x^4 + 1 - x + x^7 - \ldots\infty$$

$$= 1 - x - x^4 + x^7 + x^{13} - \ldots\infty$$

In fact we have;

$$A(x) = \prod_{n=1}^{\infty}\frac{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)\left(1-x^{5n}\right)}{\left(1-x^{5n-4}\right)^2\left(1-x^{5n-1}\right)^2},$$

$$B(x) = \prod_{n=1}^{\infty}\frac{\left(1-x^{5n}\right)}{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)},$$

$$C(x) = \prod_{n=1}^{\infty}\frac{\left(1-x^{5n}\right)}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)},$$

$$D(x) = \prod_{n=1}^{\infty}\frac{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)\left(1-x^{5n}\right)}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)}.$$

3.1 Rank of a Partition

The rank of a partition is defined as the largest part minus the number of parts. Thus the partition 6 + 5 + 2 + 1 + 1 + 1 + 1 of 17 has rank, 6–7 = –1 and the conjugated partition, 7 + 3 + 2 + 2 + 2 + 1 has rank, 7–6 = 1. i.e., the rank of a partition and that of the conjugate partition differ only in sign. The rank of a partition of 5 belongs to any one of the residues (mod 5) and we have exactly 5 residues. There is similar result for all partitions of 7 leading to (mod 7).

The generating function for the rank is of the form [3];

$$\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+|m|n}\left(1-x^n\right)\prod_{j=1}^{\infty}\left(1-x^j\right)^{-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left\{x^{\frac{n}{2}(3n+2+|m|-1)} - x^{\frac{n}{2}(3n+2|m|+1)}\right\}\sum_{k=0}^{\infty}P(k)\,x^k$$

$$= \left(x^{|m|+1} + 0.x^{|m|+2} + x^{|m|+3} + \ldots\infty\right) - \left(x^{2|m|+5} + x^{2|m|+6} + \ldots\infty\right)$$

$$= \sum_{n=0}^{\infty}N(m,n)\,x^n \cdot$$

The generating function for $N(m,t,n)$ is of the form;

$$\sum_{\substack{n=-\infty \\ n\neq 1}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\frac{\left(x^{mn} + x^{n(t-m)}\right)}{1-x^{tn}}\prod_{j=1}^{\infty}\left(1-x^j\right)^{-1}$$

$$= \sum_{\substack{n=-\infty \\ n\neq 1}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^{mn} + x^{n(t-m)}\right)\times$$

$$\left(1 + x^{tn} + x^{2tn} + \ldots\infty\right)\sum_{k=0}^{\infty}P(k)\,x^k$$

$$= \sum_{n=0}^{\infty}N(m,t,n)\,x^n ;$$

which shows that all the coefficients of x^{-n} (where n is any positive integer) are zero.

Now we define the generating function;

$r_a(d)$ for $N(a,t,tn+d)$

where $r_a(d) = r_a(d,t) = \prod_{n=0}^{\infty}N(a,t,tn+d)\,x^n$, and

$$r_{a,b}(d) = r_{a,b}(d,t) = r_a(d) - r_b(d).$$

$$= \prod_{n=0}^{\infty}\left\{N(a,t,tn+d) - N(b,t,tn+d)\right\}x^n .$$

The generating function $\phi(x)$ is of the form;

$$\phi(x) = -1 + \left\{\frac{1}{1-x} + \frac{x^5}{(1-x)\left(1-x^4\right)\left(1-x^6\right)} + \frac{x^{20}}{(1-x)\left(1-x^4\right)\left(1-x^6\right)\left(1-x^9\right)\left(1-x^{11}\right)} + \ldots\infty\right\},$$

$$= -1 + \left(1 + x + x^2 + \ldots\infty\right) + x^5\left(1 + x + x^2 + \ldots\infty\right)$$

$$\left(1+x^4+\ldots\infty\right)\left(1+x^6+\ldots\infty\right)+\ldots\infty$$

$$=x+x^2+x^3+x^4+2x^5+2x^6+2x^7+2x^8+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(1,5,5n)-N(2,5,5n)\}x^n$$

$$=r_{1,2}(0).$$

The generating function $A(x)$ is defined as;

$$A(x)=\frac{1-x^2-x^3+x^9+\ldots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\ldots\infty}$$

$$=\left(1-x^2-x^3+x^9+\ldots\infty\right)\left(1+2x+3x^2+\ldots\infty\right)$$

$$\left(1+2x^4+3x^8+\ldots\infty\right)\ldots\infty$$

$$=1+2x+2x^2+x^3+2x^4+\ldots\infty$$

$$=1+\sum_{n=0}^{\infty}\{N(0,5,5n)-N(2,5,5n)+$$

$$N(1,5,5n)-2N(2,5,5n)\}x^2$$

$$=1+\sum_{n=0}^{\infty}\{N(0,5,5n)-N(2,5,5n)\}x^n+$$

$$2\sum_{n=0}^{\infty}\{N(1,5,5n)-N(2,5,5n)\}x^n$$

$$=1+r_{0,2}(0)+2r_{1,2}(0).$$

The generating function is of the form;

$$\prod_{n=1}^{\infty}\frac{1-x^{5n}}{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)}$$

$$=\prod_{n=1}^{\infty}\left(1-x^{5n}\right)\left(1+x^{5n-4}+\ldots\infty\right)\left(1+x^{5n-1}+\ldots\infty\right)$$

$$=(1-0)+(3-2)x+(12-11)x^2+x^3+2x^4+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(0,5,5n+1)-N(2,5,5n+1)\}x^n$$

$$=r_{0,2}(1).$$

The generating function is of the form;

$$\prod_{n=1}^{\infty}\frac{1-x^{5n}}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)}$$

$$=\prod_{n=1}^{\infty}\left(1-x^{5n}\right)\left(1+x^{5n-3}+x^{10n-6}+\ldots\infty\right)$$

$$=(1-0)+(3-3)x+(16-15)x^2+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(0,5,5n+2)-N(2,5,5n+2)\}x^n\times$$

$$\left(1+x^{5n-2}+x^{10n-4}+\ldots\infty\right)$$

$$=r_{1,2}(2).$$

The generating function $\Psi(x)$ is of the form;

$$\Psi(x)=-1+\left\{\frac{1}{1-x^2}+\frac{x^5}{\left(1-x^2\right)\left(1-x^3\right)\left(1-x^7\right)}+\right.$$

$$\left.\frac{x^{20}}{\left(1-x^2\right)\left(1-x^3\right)\left(1-x^7\right)\left(1-x^8\right)\left(1-x^{12}\right)}+\ldots\infty\right\}$$

$$=-1+\left(1+x^2+x^4+\ldots\infty\right)+x^5\left(1+x^2+\ldots\infty\right)\times$$

$$\left(1+x^3+x^6+\ldots\infty\right)\left(1+x^7+\ldots\infty\right)+\ldots\infty$$

$$=x^2+x^4+x^6+x^7+2x^8+x^9+2x^{10}+\ldots\infty.$$

Hence

$$\frac{\Psi(x)}{x}=x+x^3+x^4+x^5+x^6+2x^7+x^8+2x^9+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(2,5,5n+3)-N(0,5,5n+3)\}x^n$$

$$=r_{2,0}(3)$$

and

$$r_{0,2}(3) = -\frac{\Psi(x)}{x}.$$

The generating function $D(x)$ is of the form;

$$D(x) = \frac{1 - x - x^4 + x^7 + \ldots\infty}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\ldots\infty}$$

$$= (1 - x - x^4 + x^7 + \ldots\infty)(1 + 2x^2 + 3x^4 + \ldots\infty)$$

$$(1 + 2x^3 + \ldots\infty)(1 + 2x^7 + \ldots\infty)\ldots\infty$$

$$= 1 - x + 2x^2 + 0.x^3 + \ldots\infty$$

$$= \sum_{n=0}^{\infty}\{N(0,5,5n+3) - N(1,5,5n+3) +$$

$$N(0,5,5n+3) - N(2,5,5n+3)\}x^n$$

$$= \sum_{n=0}^{\infty}\{N(0,5,5n+3) - N(1,5,5n+3)\}x^n +$$

$$\sum_{n=0}^{\infty}\{N(0,5,5n+3) - N(2,5,5n+3)\}x^n$$

$$= r_{0,1}(3) + r_{0,2}(3).$$

3.2 Mock Theta Conjectures

The generating function for $\rho_0(n)$ is of the form [5];

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(x^{n+1})_{n+1}}$$

$$= \frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \ldots\infty$$

$$= x(1 + x + x^2 + \ldots\infty) + x^3(1 + x^2 + x^4 + \ldots\infty)\times$$

$$(1 + x^3 + x^6 + \ldots\infty) + \ldots\infty$$

$$= x + x^2 + 2x^3 + x^4 + 3x^5 + 2x^7 + \ldots\infty$$

$$= \sum_{n=0}^{\infty}\rho_0(n)x^n, \qquad (4)$$

which is convenient to define $\rho_0(0) = 0$

Now we prove the Theorem, which is known as First Mock Theta Conjecture.

Theorem 1: $N(1,5,5n) = N(0,5,5n) + \rho_0(n)$, where $\rho_0(n)$ is the number of partitions of n with unique smallest part and all other parts ≤ the double of the smallest part.

Proof: From (4) we have;

$$\frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \ldots\infty$$

$$= \sum_{n=0}^{\infty}\rho_0(n)x^n$$

$$\Rightarrow x + x^2 + 2x^3 + x^4 + 3x^5 + 2x^6 + \ldots\infty$$

$$= \sum_{n=0}^{\infty}\rho_0(n)x^n$$

$$\Rightarrow 3\phi(x) + 1 - A(x) = \sum_{n=0}^{\infty}\rho_0(n)x^n \text{ (by above)}$$

$$3r_{1,2}(0) + 1 - (1 + r_{0,2}(0) + r_{1,2}(0)) = \sum_{n=0}^{\infty}\rho_0(n)x^n \text{ (by above)}$$

$$\Rightarrow r_{1,2}(0) - r_{0,2}(0) = \sum_{n=0}^{\infty}\rho_0(n)x^n \text{ (by above)}$$

$$\Rightarrow \sum_{n=0}^{\infty}\{N(1,5,5n) - N(2,5,5n) +$$

$$N(0,5,5n) - N(2,5,5n)\}x^n = \sum_{n=0}^{\infty}\rho_0(n)x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}\{N(1,5,5n) - N(0,5,5n)\}x^n = \sum_{n=0}^{\infty}\rho_0(n)x^n.$$

Equating the coefficient of x^n on both sides, we get

$N(1,5,5n) = N(0,5,5n) + \rho_0(n)$. Hence the Theorem.

The generating function for $\rho_1(n)$ is defined as;

$$\sum_{n=1}^{\infty}\frac{x^{n-1}}{\left(x^n\right)_n}$$

$$=\frac{1}{1-x}+\frac{x}{\left(1-x^2\right)\left(1-x^3\right)}+\frac{x^2}{\left(1-x^3\right)\left(1-x^4\right)\left(1-x^5\right)}+\ldots\infty$$

$$=1+2x+2x^2+3x^3+3x^4+4x^5+4x^6+6x^7+4x^8+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{\rho_1(n)+1\}x^n \text{, if we assume } \rho_1(0). \qquad (5)$$

Now we prove the Theorem, which is known as Second Mock Theta Conjecture.

Theorem 2:
$2N(2,5,5n+3)=N(1,5,5n+3)+N(0,5,5n+3)+\rho_1(n)+1$, where $\rho_1(n)$ is the number of partitions of n with unique smallest part and all other parts ≤ one plus the double of the smallest part.

Proof: We have;

$$\frac{1}{1-x}+\frac{x}{\left(1-x^2\right)\left(1-x^3\right)}+\frac{x^2}{\left(1-x^3\right)\left(1-x^4\right)\left(1-x^5\right)}+\ldots\infty$$

$$=3\frac{\Psi(x)}{x}+D(x)$$

$$\Rightarrow 1+2x+2x^2+2x^3+2x^4+3x^5+3x^6+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{\rho_1(n)+1\}x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}\{\rho_1(n)+1\}x^n=3\frac{\Psi(x)}{x}+D(x) \text{, (by above)}$$

$$3r_{2,0}(3)+r_{0,1}(3)+r_{0,2}(3)=\sum_{n=0}^{\infty}\{\rho_1(n)+1\}x^n$$

$$\Rightarrow \sum_{n=0}^{\infty}\{3N(2,5,5n+3)-3N(0,5,5n)+N(0,5,5n+3)-$$

$$N(1,5,5n+3)+N(0,5,5n+3)-N(2,5,5n+3)\}x^n$$

$$=\sum_{n=0}^{\infty}\{\rho_1(n)+1\}x^n$$

Equating the coefficient of x^n on both sides, we get;

$$2N(2,5,5n+3)-N(0,5,5n)-N(1,5,5n+3)$$

$$=\rho_1(n)+1$$

$$2N(2,5,5n+3)=N(0,5,5n)+N(1,5,5n+3)+\rho_1(n)+1.$$

Hence the Theorem.

4. Illustrative Examples

Here we give two examples, which are related to first and second mock theta conjectures respectively.

Example 1

For n = 2, we have;
$N(1,5,10)=9$ with the relevant partitions are: 8 + 2, 6 + 1 + 1 + 1 + 1, 5 + 3 + 1 + 1, 5 + 2 + 2 + 1, 4 + 4 + 2, 4 + 3 + 3, 3 + 2 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 2 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.

But $N(0,5,10)=8$ with the relevant partitions are: 8 + 1 + 1, 7 + 3, 5 + 2 + 1 + 1 + 1, 4 + 4 + 1 + 1, 4 + 3 + 2 + 1, 4 + 2 + 2 + 2, 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1 + 1 + 1.

Again, $\rho_0(2)=1$ with the relevant partition being 2.

$$N(1,5,10)=N(0,5,10)+\rho_0(2).$$

Example 2

For n = 1, we have;
$N(2,5,8)=5$ with the relevant partitions are: 8, 5 + 2 + 1, 4 + 4, 3 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1 + 1.

But N(1, 5, 8) = 4 with the relevant partitions are: 5 + 1 + 1 + 1, 4 + 3 + 1, 4 + 2 + 2, 2 + 2 + 1 + 1 + 1 + 1, and N(0, 5, 8) = 4 with the relevant partitions are: 7 + 1, 4 + 2 + 1 + 1, 3 + 3 + 2, 2 + 1 + 1 + 1 + 1 + 1 + 1.

Again $\rho_1(1)=1$ with the relevant partition being 1. Therefore, 2N(2, 5, 8) = 2×5 = 10 = 4 + 4 + 1 + 1 = $N(1,5,8)+N(0,5,8)$ + $\rho_1(1)=1$.

5. Conclusion

We have verified for any positive integer of n in two Theorems first and second mock theta conjectures. But we have seen these for n = 2 or 1 respectively.

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