Mock Theta Conjectures

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Abstract

This paper shows how to prove the two Theorems first and second mock theta conjectures respectively.

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1. Introduction

We give the definitions of \(\pi\), rank of partition, \(N(m,n)\), \(N(m,t,n)\), \(\rho_0(n)\), \(\rho_1(n)\), \(x\), \((x^2)\), \((x^5)\), which are collected from Partitions Yesterday and Today [4], Generalizations of Dyson’s Rank [3], Ramanujan’s Lost Notebook [2]. We generate the generating functions for \(\rho_0(n)\), and \(\rho_1(n)\) [2] and prove the two Theorems first and second mock theta conjectures respectively. Finally we give two numerical examples which are related to first and second mock theta conjectures respectively when \(n = 1\).

2. Definitions

\(\pi\) : A partition.

Rank of partition: The largest part of a partition \(\pi\) minus the number of parts of \(\pi\).

\(N(m,n)\) : The number of partitions of \(n\) with rank \(m\).

\(N(m,t,n)\) : The number of partitions of \(n\) with rank congruent to \(m\) modulo \(t\).

\(\rho_0(n)\) : The number of partitions of \(n\) with unique smallest part and all other parts \(\leq\) the double of the smallest part.

\(\rho_1(n)\) : The number of partitions of \(n\) with unique smallest part and all other parts \(\leq\) one plus the double of the smallest part.

\(x\) : The set of complex numbers.

3. Mock Theta Functions (2)

We quote the relations below [1, 2]:

\[ F(x) = \frac{(1-x)(1-x^2)(1-x^4)\ldots}{(1-2x \cos \frac{2n\pi}{5} + x^2)(1-2x^2 \cos \frac{2n\pi}{5} + x^4)\ldots} \]

\[ f'(x) = 1 + \frac{x}{1-2x \cos \frac{2n\pi}{5} + x^2} + \frac{x^4}{(1-2x \cos \frac{2n\pi}{5} + x^2)(1-2x^2 \cos \frac{2n\pi}{5} + x^4)} \ldots \]

\(n = 1\) or \(2\).
\[ F(x^3) = A(x) - 4x^3 \cos \frac{2n\pi}{5} B(x) + 2x^2 \cos \frac{4n\pi}{5} C(x) - \]

\[ 2x^3 \cos \frac{2n\pi}{5} D(x). \]  

(1)

\[ f'(x^3) = \left\{ A(x) - 4 \sin \frac{2n\pi}{5} \Phi(x) \right\} + \frac{x}{2} B(x) + 2x^2 \cos \frac{2n\pi}{5} C(x) - \]

\[ 2x^3 \cos \frac{2n\pi}{5} \left\{ D(x) + 4 \sin \frac{2n\pi}{5} \Psi(x) \right\}. \]  

(2)

\[ A(x) = \frac{1 - x^3 - x^3 + \ldots}{(1 - x^2)(1 - x^3)(1 - x^6)^2 \ldots} \]

\[ B(x) = \frac{(1 - x^3)(1 - x^{10})(1 - x^{15}) + \ldots}{(1 - x)(1 - x^2)(1 - x^6)^2 \ldots} \]

\[ C(x) = \frac{(1 - x^3)(1 - x^{10})(1 - x^{15}) + \ldots}{(1 - x^2)(1 - x^3)(1 - x^6)^2 \ldots} \]

\[ D(x) = \frac{1 - x - x^3 + x^3 + \ldots}{(1 - x^2)(1 - x^3)(1 - x^6)^2 \ldots} \]

\[ \phi(x) = -1 + \frac{\frac{x^5}{(1 - x)(1 - x^4)(1 - x^6)^2 \ldots}}{(1 - x^2)(1 - x^3)(1 - x^6)^2 \ldots} \]

But we get;

\[ A(x^3) - 4x \cos \frac{2\pi}{5} B(x^3) + 2x^2 \cos \frac{4\pi}{5} C(x^3) - \]

\[ 2x^3 \cos \frac{2\pi}{5} D(x^5) = 1 - 4x \cos \frac{2\pi}{5} + 2x^2 \cos \frac{4\pi}{5} - 2x^3 \cos \frac{2\pi}{5} + 2x^5 - \]

\[ 4x^6 \cos \frac{2\pi}{5} + 2x^8 \cos \frac{2\pi}{5} + 2x^{10} + \ldots \]

\[ \Psi(x) = -1 + \frac{\frac{x^5}{(1 - x^2)(1 - x^4)(1 - x^6)^2 \ldots}}{(1 - x^2)(1 - x^3)(1 - x^6)^2 \ldots} \]

Now,

\[ \frac{x}{1 - x} + \frac{x^3}{(1 - x^2)(1 - x^6)^2 \ldots} + \frac{x^5}{(1 - x^2)(1 - x^4)(1 - x^6)^2 \ldots} + \ldots \infty \]

\[ = 3\phi(x) + 1 - A(x). \]

And,

\[ \frac{x}{1 - x} + \frac{x^2}{(1 - x^2)(1 - x^6)^2 \ldots} + \frac{x^3}{(1 - x^2)(1 - x^4)(1 - x^6)^2 \ldots} + \ldots \infty \]

\[ = 3\Psi(x) + xD(x). \]

We assume without loss of generality that \( n = 1 \). Let \( \zeta = \exp \frac{2\pi i}{3} \), then we may write the definitions of \( F(x) \) and \( f'(x) \) as;

\[ F(x) = \frac{\phi(x)}{(\zeta x)(\zeta^{-1} x)} \]

and

\[ f'(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{n^2}}{(1 - \zeta x)(1 - \zeta^{-1} x)(1 - \zeta^2 x)(1 - \zeta^{-2} x)} \]

\[ = \sum_{n=1}^{\infty} \frac{x^{n^2}}{(\zeta x)(\zeta^{-1} x)} \]

where we have used the relations;

\[ (a)_0 = 1, \ (a)_n = (1 - a)(1 - ax) \ldots (1 - ax^{n-1}), \quad \text{for} \ n \geq 1 \]

and

\[ (a)_n = \lim_{x \to 1} (1 - ax^{n-1}). \]

After replacing \( x \) by \( x^3 \) we see that (1) and (2) are identities for \( F(x) \) and \( f'(x) \). We note that the numerators in the definitions of \( A(x) \) and \( D(x) \) are theta series in \( x \) and hence may be written as infinite products using Jacobi’s triple product identity;

\[ \prod_{n=1}^{\infty} (1 - \zeta x^n)(1 - \zeta^{-1} x^n)(1 - \zeta^2 x^n) \]

\[ = \prod_{n=1}^{\infty} (1 - x^{n^2}) \]

(3)
\[= \cdots + z^{-2}x - z^{-1} + 1 - zx + z^{-2}x^3 - \cdots \infty.\]

where \(z \neq 0\) and \(|x| < 1\).

Replacing \(x\) by \(x^5\) and \(z\) by \(x^{-3}\) we get from (3);

\[
\prod_{n=1}^{\infty} \left( 1 - x^{5n-3} \right) \left( 1 - x^{5n-2} \right) \left( 1 - x^{5n-1} \right) \\
= \cdots + x^{11} + 1 - x^2 + x^9 - \cdots \infty \\
= 1 - x^2 - x^3 + x^9 + x^{11} - \cdots \infty.
\]

Again replacing \(x\) by \(x^5\) and \(z\) by \(x^{-3}\) (3) becomes;

\[
\prod_{n=1}^{\infty} \left( 1 - x^{5n-3} \right) \left( 1 - x^{5n-2} \right) \left( 1 - x^{5n-1} \right) \\
= \cdots + x^{13} - x^4 + 1 - x^2 - x^9 - \cdots \infty \\
= 1 - x - x^4 + x^7 + x^{13} - \cdots \infty.
\]

In fact we have;

\[
A(x) = \prod_{n=1}^{\infty} \frac{\left( 1 - x^{5n-3} \right) \left( 1 - x^{5n-2} \right) \left( 1 - x^{5n-1} \right)}{\left( 1 - x^{5n-4} \right)^2 \left( 1 - x^{5n-1} \right)^2},
\]

\[
B(x) = \prod_{n=1}^{\infty} \frac{\left( 1 - x^{5n-3} \right)}{\left( 1 - x^{5n-4} \right) \left( 1 - x^{5n-1} \right)},
\]

\[
C(x) = \prod_{n=1}^{\infty} \frac{\left( 1 - x^{5n} \right)}{\left( 1 - x^{5n-2} \right) \left( 1 - x^{5n-3} \right)},
\]

\[
D(x) = \prod_{n=1}^{\infty} \frac{\left( 1 - x^{5n} \right) \left( 1 - x^{5n-3} \right) \left( 1 - x^{5n-1} \right)}{\left( 1 - x^{5n-2} \right) \left( 1 - x^{5n-4} \right)}.
\]

3.1 Rank of a Partition

The rank of a partition is defined as the largest part minus the number of parts. Thus the partition \(6 + 5 + 2 + 1 + 1 + 1 + 1\) of 17 has rank, \(6-7 = -1\) and the conjugated partition, \(7 + 3 + 2 + 2 + 1\) has rank, \(7-6 = 1\), i.e., the rank of a partition and that of the conjugate partition differ only in sign. The rank of a partition of 5 belongs to any one of the residues (mod 5) and we have exactly 5 residues. There is similar result for all partitions of 7 leading to (mod 7).

The generating function for the rank is of the form [3];

\[
\sum_{n=1}^{\infty} (-1)^{n-1} x^{\frac{\phi(n)-1}{\phi(n)}} \prod_{j=1}^{\infty} (1-x^j)^{-1}
\]

\[= \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ x^{\frac{\phi(n)+1}{\phi(n)}} - x^{\frac{\phi(n)+2}{\phi(n)+1}} \right\} \sum_{k=0}^{\infty} P(k) x^k
\]

\[= \left( x^{\phi(1) + 1} + x^{\phi(2) + 1} + x^{\phi(3) + 1} + \cdots \infty \right) - \left( x^{\phi(5) + 1} + x^{\phi(6) + 1} + \cdots \infty \right)
\]

\[= \sum_{n=0}^{\infty} N(m,n) x^n.
\]

The generating function for \(N(m, t, n)\) is of the form;

\[
\sum_{n=0}^{\infty} (-1)^{n} x^{\frac{\phi(n)+1}{\phi(n)}} \prod_{j=1}^{\infty} \left( 1 - x^j \right)^{-1}
\]

\[= \sum_{n=0}^{\infty} (-1)^{n} x^{\frac{\phi(n)+1}{\phi(n)}} \left( x^{\phi(n)+1} + x^{\phi(n)+2} + \cdots \infty \right) \sum_{k=0}^{\infty} P(k) x^k
\]

\[= \sum_{n=0}^{\infty} N(m, t, n) x^n;
\]

which shows that all the coefficients of \(X^{-n}\) (where \(n\) is any positive integer) are zero.

Now we define the generating function;

\[r_u(d)\] for \(N(a, t, m+d)\)

where \(r_u(d) = r_u(d, t) = \prod_{n=0}^{\infty} N(a, t, m+d) x^n\), and

\[r_u, b(d) = r_u, b(d, t) = r_u(d) - r_b(d);
\]

\[= \prod_{n=0}^{\infty} \left\{ N(a, t, m+d) - N(b, t, m+d) \right\} x^n.
\]

The generating function \(\phi(x)\) is of the form;

\[
\phi(x) = -1 + \left\{ \frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \frac{x^{2n}}{(1-x)(1-x^4)(1-x^6)(1-x^{11} + \cdots \infty)} \right\}
\]

\[= -1 + \left( 1 + x + x^2 + \cdots \infty \right) + x^5 \left( 1 + x + x^2 + \cdots \infty \right)
\]
\[(1 + x^4 + \ldots \infty)(1 + x^6 + \ldots \infty) + \ldots \infty = x + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + \ldots \infty\]

\[= \sum_{n=0}^{\infty} \{N(1,5,5n) - N(2,5,5n)\} x^n = r_{1,2}(0)\]

The generating function \(A(x)\) is defined as:

\[A(x) = \frac{1 - x^2 - x^3 + x^9 + \ldots \infty}{(1 - x)^2(1 - x^2)^2(1 - x^6)^2 \ldots \infty}\]

\[= (1 - x^2 - x^3 + x^9 + \ldots \infty)(1 + 2x + 3x^2 + \ldots \infty)\]

\[= (1 + 2x + 2x^2 + x^3 + 2x^4 + \ldots \infty) = 1 + \sum_{n=0}^{\infty} \{N(0,5,5n) - N(2,5,5n)\} + N(1,5,5n) - 2N(2,5,5n)\]

\[= 1 + \sum_{n=0}^{\infty} \{N(0,5,5n) - N(2,5,5n)\} x^n + 2\sum_{n=0}^{\infty} \{N(1,5,5n) - N(2,5,5n)\} x^n = 1 + r_{0,2}(0) + 2r_{1,2}(0)\]

The generating function is of the form:

\[\prod_{n=4}^{\infty} \frac{1 - x^{5n}}{(1 - x^{5n-3})(1 - x^{5n-2})} = \prod_{n=4}^{\infty} (1 - x^{5n})(1 + x^{5n-3} + x^{10n-6} + \ldots \infty) = (1 - 0) + (3 - 3)x + (16 - 15)x^2 + \ldots \infty\]

\[= \sum_{n=0}^{\infty} \{N(0,5,5n + 2) - N(2,5,5n + 2)\} x^n + (1 + x^{5n-2} + x^{10n-4} + \ldots \infty) = r_{1,2}(2)\]

The generating function \(\Psi(x)\) is of the form:

\[\Psi(x) = -1 + \left\{ \frac{1}{(1 - x^2)(1 - x^3)(1 - x^4)} + \frac{x^5}{(1 - x^2)(1 - x^3)(1 - x^4)} \right\} + \ldots \infty\]

\[= -1 + \left( 1 + x^2 + x^4 + \ldots \infty \right) + x^5 \left( 1 + x^2 + \ldots \infty \right) \times \left( 1 + x^3 + x^6 + \ldots \infty \right) \times \left( 1 + x^7 + \ldots \infty \right) + \ldots \infty\]

\[= x^2 + x^4 + x^6 + x^7 + 2x^8 + x^9 + 2x^{10} + \ldots \infty\]

Hence

\[\Psi(x) = \frac{x}{x} = x + x^3 + x^4 + x^5 + x^6 + 2x^7 + x^8 + 2x^9 + \ldots \infty\]

\[= \sum_{n=0}^{\infty} \{N(2,5,5n + 3) - N(0,5,5n + 3)\} x^n = r_{2,0}(3)\]

and
\[ r_{0,2}(3) = -\frac{\Psi(x)}{x}. \]

The generating function \( D(x) \) is of the form:
\[
D(x) = \frac{1 - x - x^4 + x^7 + \ldots}{(1 - x^2)^2 (1 - x^3)^2 (1 - x^7)^2 \ldots} \\
= (1 - x - x^4 + x^7 + \ldots)(1 + 2x^2 + 3x^4 + \ldots) \\
(1 + 2x^3 + \ldots)(1 + 2x^7 + \ldots) \\
= 1 - x + 2x^2 + 0x^3 + \ldots
\]
\[
= \sum_{n=0}^{\infty} \{N(0,5,5n + 3) - N(1,5,5n + 3) + N(0,5,5n + 3) - N(2,5,5n + 3)\} x^n
\]
\[
= \sum_{n=0}^{\infty} \{N(0,5,5n + 3) - N(1,5,5n + 3)\} x^n + \sum_{n=0}^{\infty} \{N(0,5,5n + 3) - N(2,5,5n + 3)\} x^n
\]
\[
= r_{0,1}(3) + r_{0,2}(3).
\]

### 3.2 Mock Theta Conjectures

The generating function for \( \rho_n(n) \) is of the form [5]:
\[
\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x^{2n+1})_{n+1}} = \frac{x}{1 - x + \frac{x^3}{(1-x^2)(1-x^3)(1-x^5)}} + \frac{x^5}{(1-x^2)(1-x^3)(1-x^7)} + \ldots
\]
\[
= x(1 + x + x^2 + \ldots) + x^5(1 + x^2 + x^4 + \ldots) + \ldots
\]
\[
= x + x^2 + 2x^3 + x^4 + 3x^5 + 2x^7 + \ldots
\]
\[
= \sum_{n=0}^{\infty} \rho_n(n) x^n.
\]

which is convenient to define \( \rho_0(0) = 0 \)

Now we prove the Theorem, which is known as First Mock Theta Conjecture.

**Theorem 1:** \( N(1,5,5n) = N(0,5,5n) + \rho_n(n) \), where \( \rho_n(n) \) is the number of partitions of \( n \) with unique smallest part and all other parts \( \leq \) the double of the smallest part.

**Proof:** From (4) we have:
\[
\frac{x}{1 - x + \frac{x^3}{(1-x^2)(1-x^3)(1-x^5)}} + \frac{x^5}{(1-x^2)(1-x^3)(1-x^7)} + \ldots
\]
\[
= \sum_{n=0}^{\infty} \rho_n(n) x^n
\]
\[
\Rightarrow x + x^2 + 2x^3 + x^4 + 3x^5 + 2x^7 + \ldots
\]
\[
= \sum_{n=0}^{\infty} \rho_n(n) x^n
\]
\[
\Rightarrow 3\phi(x) + 1 - A(x) = \sum_{n=0}^{\infty} \rho_n(n) x^n \quad \text{(by above)}
\]
\[
3r_{1,2}(0) + 1 - (1 + r_{1,2}(0) + r_{1,2}(0)) = \sum_{n=0}^{\infty} \rho_n(n) x^n \quad \text{(by above)}
\]
\[
\Rightarrow r_{1,2}(0) - r_{0,2}(0) = \sum_{n=0}^{\infty} \rho_n(n) x^n \quad \text{(by above)}
\]
\[
\Rightarrow \sum_{n=0}^{\infty} \{N(1,5,5n) - N(2,5,5n)\} x^n
\]
\[
= \sum_{n=0}^{\infty} \{N(1,5,5n) - N(0,5,5n)\} x^n
\]

Equating the coefficient of \( x^n \) on both sides, we get
\[
N(1,5,5n) = N(0,5,5n) + \rho_n(n).
\]

Hence the Theorem.

The generating function for \( \rho_1(n) \) is defined as;
\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{1-x} + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^3} + \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} + \ldots
\]

\[
= 1 + 2x + 2x^2 + 3x^3 + 3x^4 + 4x^5 + 6x^7 + 4x^8 + \ldots
\]

\[
= \sum_{n=0}^{\infty} \{\rho_1(n) + 1\} x^n, \text{ if we assume } \rho_1(0).
\]  

(5)

Now we prove the Theorem, which is known as Second Mock Theta Conjecture.

**Theorem 2:**

\[
2N(2,5,5n+3) = N(1,5,5n+3) + N(0,5,5n+3) + \rho_1(n)+1,
\]

where \( \rho_1(n) \) is the number of partitions of \( n \) with unique smallest part and all other parts \( \leq \) one plus the double of the smallest part.

**Proof:** We have;

\[
1 - \frac{x}{1-x} + \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3} + \frac{x^4}{(1-x)^4} + \frac{x^5}{(1-x)^5} + \ldots
\]

\[
= 3 \frac{\psi(x)}{x} + D(x)
\]

\[
\Rightarrow 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 3x^5 + 3x^6 + \ldots
\]

\[
= \sum_{n=0}^{\infty} \{\rho_1(n) + 1\} x^n
\]

\[
= \sum_{n=0}^{\infty} \{\rho_1(n) + 1\} x^n = 3 \frac{\psi(x)}{x} + D(x), \text{ (by above)}
\]

\[
3r_{2,0}(3) + r_{1,1}(3) + r_{0,1}(3) = \sum_{n=0}^{\infty} \{\rho_1(n) + 1\} x^n
\]

\[
\Rightarrow \sum_{n=0}^{\infty} \{3N(2,5,5n+3) - 3N(0,5,5n) + N(0,5,5n+3) - N(1,5,5n+3)\} x^n
\]

\[
= \sum_{n=0}^{\infty} \{\rho_1(n) + 1\} x^n
\]

Equating the coefficient of \( x^n \) on both sides, we get;

\[
2N(2,5,5n+3) - N(0,5,5n) - N(1,5,5n+3) = \rho_1(n) + 1
\]

\[
2N(2,5,5n+3) = N(0,5,5n) + N(1,5,5n+3) + \rho_1(n)+1.
\]

Hence the Theorem.

**4. Illustrative Examples**

Here we give two examples, which are related to first and second mock theta conjectures respectively.

**Example 1**

For \( n = 2 \), we have;

\[ N(1,5,10) = 9 \text{ with the relevant partitions are: } 8 + 2 + 6 + 1 + 1 + 1, 5 + 3 + 1 + 1, 5 + 2 + 2 + 1, 4 + 4 + 2, 4 + 3 + 3, 3 + 2 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1.
\]

But \( N(0,5,10) = 8 \) with the relevant partitions are:

\[ 8 + 1 + 1, 7 + 3, 5 + 2 + 1 + 1, 4 + 4 + 1 + 1, 4 + 3 + 3, 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1.
\]

At \( N(1,5,10) = 9 \) the relevant partition being 2.

\[ N(1,5,10) = N(0,5,10) + \rho_1(2). \]

**Example 2**

For \( n = 1 \), we have;

\[ N(2,5,8) = 5 \text{ with the relevant partitions are: } 8, 5 + 2 + 1, 4 + 4, 3 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1 + 1.
\]

But \( N(1,5,8) = 4 \) with the relevant partitions are:

\[ 7 + 1, 4 + 2 + 1 + 1 + 1, 3 + 3 + 2, 2 + 1 + 1 + 1 + 1 + 1 + 1.
\]

Again \( \rho_1(2) = 1 \) with the relevant partition being 1.

Therefore, \( 2N(2,5,8) = 2 \times 5 = 10 = 4 + 4 + 1 + 1 + 1 = N(1,5,8) + N(0,5,8) + \rho_1(1) \).

**5. Conclusion**

We have verified for any positive integer of \( n \) in two Theorems first and second mock theta conjectures. But we have seen these for \( n = 2 \) or 1 respectively.

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